Intrinsically correct & total functional programming using locally contractive functors

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Overview

This talk is about defining **recursive algorithms** in functional languages such that they are **intrinsically correct & total**

- Following the paradigm of **structured recursion**, the recursion behaviour of **divide-and-conquer** algorithms can be separated from their recursive steps, which can be expressed categorically as **(co)algebras** for a **functor**
- We show how functors for intrinsically correct functional algorithms can be expressed as being **locally contractive** (Birkedal, Møgelberg, Schwinghammer, and Støvring 2012)
- these algorithms are then **intrinsically total**
- for recursion this means **termination**, for corecursion **productivity**.

QuickSort as a Coalgebra-to-Algebra Morphism[1]

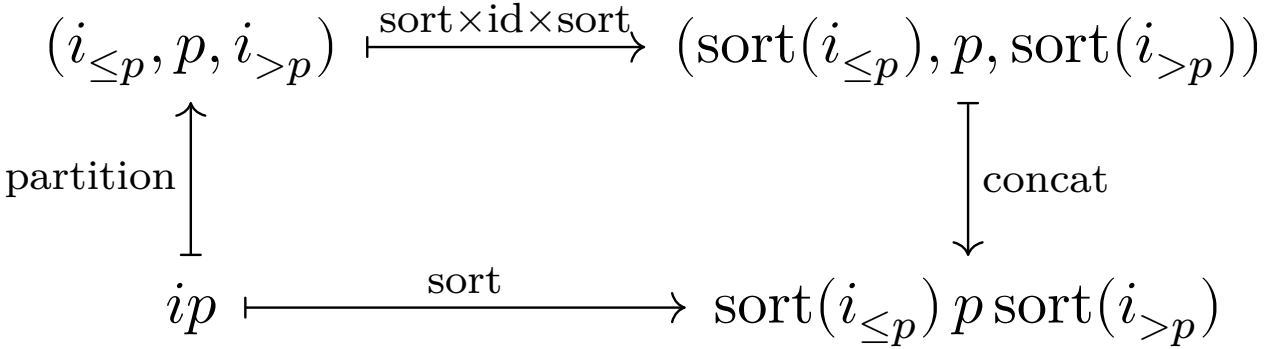


Figure: quicksort recursive case

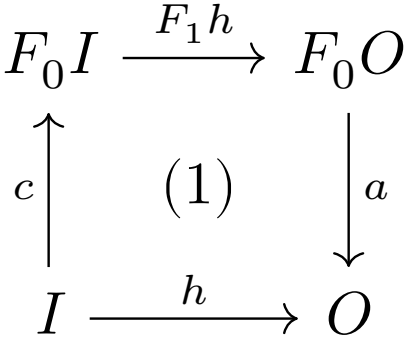


Figure: c2a-morphism

- for partition: $F := B$ for $BX := 1 + X \times A \times X$, $\quad I := A^*$.
- A c2a-morphism h is called a **hylomorphism** if it is the unique c2a-morphism making (1) commute, i.e. (1) functions as a **definition**.
- NB: In a non-total programming language, `h := c ; F h ; a` **is already** a definition. **Goal**: recover a similar definition scheme in a **total** language.

[1] due to Capretta, Uustalu, and Vene 2006

Pars pro Toto : Total from Partial Correctness

- The type $A^* \to 1 + A^* \times A \times A^*$ for $\mathrm{partition}$ is **too coarse!**
- As Alexandru, Choudhury, Rot, and Weide 2025: Move to $\mathrm{Set}/\mathcal{M}A$

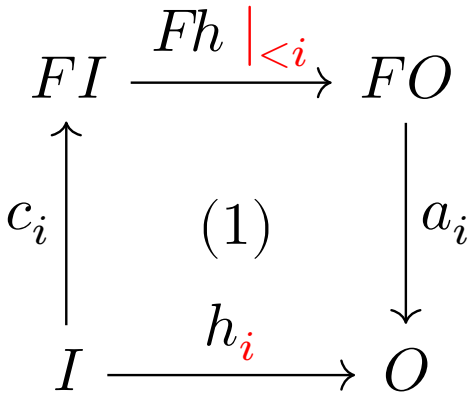
$$\overline{B}\begin{pmatrix} X \\ f \end{pmatrix} := \begin{pmatrix} 1 \\ * \mapsto \emptyset \end{pmatrix} + \begin{pmatrix} \{(l,p,r) \in X \times A \times X \mid \ldots\} \\ (l,p,r) \mapsto f(l) \uplus \{p\} \uplus f(r) \end{pmatrix}$$

$$\begin{array}{ccc} ip & \xmapsto{\quad\mathrm{partition}\quad} & (i_{\leq p}, p, i_{>p}) \\ \Big\downarrow \mathrm{toBag} & & \Big\downarrow \mathrm{toBag} \otimes \{-\} \otimes \mathrm{toBag} \\ \mathrm{toBag}(i) \uplus \{p\} & = & \mathrm{toBag}(i_{\leq p}) \uplus \{p\} \uplus \mathrm{toBag}(i_{>p}) \end{array}$$

- Note: The multiset indices in functorial positions are smaller than the input index: $f(l), f(r) < f(l) \uplus \{p\} \uplus f(r)$

Approach

- Observation: "indices in functorial positions are smaller than the input index"



$$h_i = c_i ; (F(h|_{<i}))_i ; a_i$$

- $\Rightarrow$ we could construct a solution to (1) "as an inhabitant of the homset using well founded recursion"!
- Next: How this is **made formal**. We begin with moving from Hom-**sets** to Hom-**objs** for some category which exposes structure to construct objects using well founded recursion

Plan

- **Enrichment of a Category**: Give hom objects the structure we need to do ...
- **Guarded recursion**: abstract interface to construct hom objects using well founded recursion.
- **Enrichment of a Functor**: Notion need to express ...
- **Locally Contractive Functors**: Functors which give rise to hylomorphisms via guarded recursion.

Enriched Categories & Enrichment

Fix a set of objects C_0 and a monoidal category $(V, \otimes, \mathbb{1})$.

Definition (Enriched category)

- for every $X, Y \colon C_0$ the internal Hom $(X \Rightarrow Y) : V$
- *... I will only show parts of definitions in this talk*

- Denote with $\Gamma X := \mathbb{1} \to X; \Gamma : V \to \mathrm{Set}$. Then for enriched C' there is the **underlying** plain (i.e. Set-enriched) category $(\Gamma^* C', \Rightarrow')$ with $X \Rightarrow' Y := \mathbb{1} \to (X \Rightarrow Y) = \Gamma\,(X \Rightarrow Y)$

Definition (Enrichment of a category)

- An enriched Category for C_0, with
- $C \cong \Gamma C'$, in particular $\lceil - \rceil : X \to Y \cong \Gamma\,(X \Rightarrow Y)$

Families enriched in Families

- The Category of Families (Set^A) is Cartesian monodial, inheriting that structure pointwise from Set
- In particular, $\mathbb{1} = \forall i \in A.\, \mathbb{1}_{\mathrm{Set}}$
- The external hom is $X \to Y = \forall i \in A.\, Xi \to Yi$
- We consider the enrichment of families in families:
  - The internal hom: $(X \Rightarrow Y) : \mathrm{Set}^A; (X \Rightarrow Y)i = (X \to Y)i$
  - External & internal homs agree: $\lceil - \rceil : X \to Y \cong \mathbb{1} \to (X \Rightarrow Y)$.

A Model of Guarded Recursive Terms

Definition

Given:

- A Category C enriched in Monoidal Category E
- An endofunctor $\blacktriangleright : E \to E$
- A nat. trans. $\mathrm{next} : \mathrm{Id} \to \blacktriangleright$

E is a model of guarded recursive terms if:

$$\forall H : \blacktriangleright(X \Rightarrow Y) \to (X \Rightarrow Y).\ \exists!\, \mathrm{L\ddot{o}b} H : \Gamma\,(X \Rightarrow Y)\,.$$
$$\mathrm{L\ddot{o}b} H = \mathrm{L\ddot{o}b} H ; \mathrm{next} ; H$$

In particular: $\lceil \mathrm{L\ddot{o}b} H \rceil : X \to Y$. I.e. $\mathrm{L\ddot{o}b}$ gives a guarded fixpoint for internal homs.

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E is a model of guarded recursive terms if:

$$\forall H : \blacktriangleright(\quad P \quad) \to (\quad P \quad).\ \exists!\, \mathrm{Löb} H : \Gamma\,(\quad P \quad).$$
$$\mathrm{Löb} H = \mathrm{Löb} H ; \mathrm{next} ; H$$

$\mathrm{PSh}(A, <)$ as a model of guarded recursive terms

- $\blacktriangleright : \mathrm{PSh}(A, <) \to \mathrm{PSh}(A, <)$
  $\blacktriangleright Pi = \forall j < i.\, Pj$
- $\mathrm{next} : \mathrm{Id} \to \blacktriangleright$
  $\mathrm{next}\, i : Pi \to \forall (p : j < i).\, Pj$ given by the presheaf's restriction maps.
- Löb given by well founded recursion:
  $\forall H : \blacktriangleright P \to P.\ \exists!\, \mathrm{Löb} H : \mathbb{1} \to P.\ \mathrm{Löb} H = \mathrm{Löb} H ; \mathrm{next} ; H$
  $(\forall i.\, (\forall j < i.\, Pj) \to Pi) \to \forall i.\, Pi$

Change of Base of Enrichment

Let $(M, *, I)$ and $(N, \otimes, J)$ be monoidal categories.

Definition

- A **lax monoidal** functor $F\colon M \to N$ has a comparison arrow $\varepsilon\colon J \to FI$.
- It **preserves underlying categories** if $\hat{\varepsilon}_X := \varepsilon; F_1{-}\colon M(I, X) \to N(J, FX)$ is a bijection.

Proposition (Change of Base)

Let $(C', \Rightarrow)$ be an M-enriched category for C_0.

- There is an N-enriched category $(F^*C', \Rightarrow')$ for C_0 called **the base-change of** C' **along** F with $X \Rightarrow' Y := F(X \Rightarrow Y)$.
- If F preserves underlying categories, it induces a change of base of **enrichment**. For $(C', \lceil - \rceil)$ set $(F^*(C'), \lceil - \rceil; \hat{\varepsilon})$.

Families Enriched in Presheaves

- We already saw families enriched in families $(\mathrm{Fam}(A), \Rightarrow)$
- To change their base of enrichment to presheaves, we need a lax monoidal functor $\mathrm{Fam}(A) \to \mathrm{PSh}(A)$ that preserves underlying categories.
- Enter the **cofree presheaf functor**: $U \dashv \mathrm{Cofree}$, $\mathrm{Cofree}\, Xi := \forall j \leq i.\, Xj$
- lax monoidal structure & preservation of underlying categories derives from RAPL
- Then enrichment of families in presheaves is given by $(\mathrm{Cofree}^*(\mathrm{Fam}(A)), \Rightarrow')$ with $X \Rightarrow' Y = \mathrm{Cofree}\,(X \Rightarrow Y)$

Enriched Functors & Enrichment

Definition (Enriched functor)

An **enriched functor** between enriched categories $(C', \Rightarrow)$, $(D', \Rightarrow')$ for C_0, D_0, and a map $F_0 \colon C_0 \to D_0$, has an internal functorial action $F_1' \colon X \Rightarrow Y \to F_0 X \Rightarrow' F_0 Y'$.

Definition (Enrichment of a functor)

Given categories C, D with enrichments $(\lceil - \rceil)$ and $(\lceil - \rceil')$, an **enrichment** of a functor $F \colon C \to D$ is an enriched functor for F_0 such that F_1 and F_1' **agree**: $\forall f : X \to Y.\ \lceil F_1 f \rceil' = \lceil f \rceil ; F_1'$.

Locally Contractive Functors

Definition

Given:

- V a model of guarded recursive terms
- Categories C, D with V-enrichments $(C', \Rightarrow, \lceil - \rceil), (D', \Rightarrow' \lceil - \rceil')$
- Functor $F\colon C \to D$
- An enriched functor $F'\colon \blacktriangleright^* C' \to D'$ for F_0 (its **contractive action**)

F is **locally contractive** if it has as enrichment the sequence of enrichments $C' \xrightarrow{\text{next}} \blacktriangleright^* C' \xrightarrow{F'} D'$, i.e.

$$\forall f\colon X \to Y.\ \lceil F_1 f \rceil' = \lceil f \rceil ; \text{next} ; F_1'$$

lcf concretely for Presheaf enrichment

- $f \colon X \to Y$
- $\lceil f \rceil ; \mathrm{next}\, i * \ \in \blacktriangleright (X \Rightarrow Y) i = \forall j < i.\, (X \Rightarrow Y) j$
- $\lceil f \rceil ; \mathrm{next}\, i * \ = \lceil f \rceil \mid_{<i}$
- So "$\lceil F_1 f \rceil = \lceil f \rceil ; \mathrm{next} ; F_1'$" says "$F_1'$ is a way to apply F to Homs pointwise restricted to lower indices"

Locally Contractive Functors give Hylomorphisms

$$\begin{array}{ccc} F_0 I & \xrightarrow{1?} & F_0 O \\ {\scriptstyle c}\uparrow & (1) & \downarrow{\scriptstyle a} \\ I & \xrightarrow{0?} & O \end{array}$$

- Note that in (1), the functional $H(1?) := c; 1?; a$ has type $(F_0 I \Rightarrow F_0 O) \to (I \Rightarrow O)$.
- Local contractivity of F gives us $F_1' : \blacktriangleright(I \Rightarrow O) \to (F_0 I \Rightarrow F_0 O)$.
- Putting them together we get

$$\blacktriangleright(I \Rightarrow O) \xrightarrow{F_1'} (F_0 I \Rightarrow F_0 O) \xrightarrow{H} (I \Rightarrow O).$$

- Applying $\mathrm{Löb} : (\blacktriangleright(I \Rightarrow O) \to (I \Rightarrow O)) \to \Gamma\,(I \Rightarrow O)$ gives us

$$\lceil \mathrm{Löb}(F_1'; H) \rceil : (I \to O).$$

There and Back Again : Families are Enough

Bringing local contractivity to Families by transposition along $U \dashv \mathrm{Cofree}$

- We can define a version of $\blacktriangleright$ for families by $\blacktriangleright' := U \circ \blacktriangleright \circ \mathrm{Cofree}$
- for lcf, the contractive action of F lives in
  $\blacktriangleright(X \Rightarrow' Y) \to (FX \Rightarrow' FY) =$

  $\mathrm{Psh}(A)\,(\blacktriangleright \mathrm{Cofree}\,(X \Rightarrow Y), \mathrm{Cofree}\,(FX \Rightarrow FY)) =$

  $\mathrm{Psh}(A)\,\left(\blacktriangleright \mathrm{Cofree}\,(X \Rightarrow Y), \mathrm{Cofree}\,FX \Rightarrow FY\right)$

  $\cong \langle$ transpose this homset along $U \dashv \mathrm{Cofree}$! $\rangle$

  $\mathrm{Fam}(A)\,\left(U\,\blacktriangleright \mathrm{Cofree}\,(X \Rightarrow Y), FX \Rightarrow FY\right) =$

  $\blacktriangleright'(X \Rightarrow Y) \to (FX \Rightarrow FY)$

The Prestige: $\overline{B}$ is locally contractive

$$\overline{B}\begin{pmatrix} X \\ f \end{pmatrix} := \begin{pmatrix} 1 \\ * \mapsto \emptyset \end{pmatrix} + \begin{pmatrix} \{(l,p,r) \in X \times A \times X \mid \ldots\} \\ (l,p,r) \mapsto f(l) \uplus \{p\} \uplus f(r) \end{pmatrix}$$

$$\blacktriangleright'(X \Rightarrow Y) \to (\overline{B}X \Rightarrow \overline{B}Y) = \forall i.\, (\forall j < i.\, Xj \to Yj) \to (\overline{B}Xi \to \overline{B}Yi)$$

$$\begin{aligned}
&\overline{B}' \ : \forall i.\, (\forall j < i.\, Xj \to Yj) \to (\overline{B}Xi \to \overline{B}Yi) \\
&\overline{B}' \ (i = \emptyset) \qquad\qquad f \ * \qquad\qquad := * \\
&\overline{B}' \ (i = j \uplus \{p\} \uplus k) \ f \ (l \in Xj, p, r \in Xk) := (fl, p, fr)
\end{aligned}$$

Conclusion

- Novel application of locally contractive functors to total functional programming
- Can be used "behind the scenes" as a theoretical underpinning while the programmer need only work in the setting of indexed datatypes (families)
- Illustrates an interesting novel usecase for lcf in a category which is not self-enriched

TODOs

- Use composition theorems to reduce client obligations: Famously local contractivity as is closed under composition with enriched functors
- See whether solving domain equations using lcf is also useful in this setting