Intrinsically Correct Sorting in Cubical Agda

Cass Alexandru[1] Vikraman Choudhury[2] Jurriaan Rot[3]
Niels van der Weide[3]

[1]RPTU Kaiserslautern-Landau

[2]University of Bologna & INRIA OLAS

[3]Radboud University Nijmegen

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Motivation

- "Sorting with Bialgebras and Distributive Laws" (Hinze et al. 2012)

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- Contribution: intrinsic verification of business logics & setting in which correctness of the dual algorithms follows
- Key idea: Index data by the multiset of their elements

Outline

1 Sorting as an Index-Preserving Map

2 Recap of "Sorting with Bialgebras and Distributive Laws"
- Base Functors
- Bialgebraic Semantics

3 Correct Sorting using Distributive Laws
- Base Functors for Element-Indexed (Ordered) Lists
- The FMSet Index as a Termination Measure

4 Conclusion & Future Work

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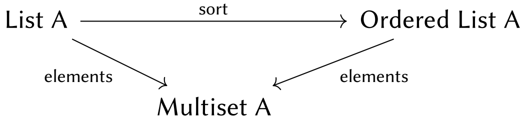
- sort : List A → List A ?
- sort : List A → Ordered List A?
- "The output should be a permutation of the input"
  - "Mapping a list to the multiset of its elements is invariant under sorting"



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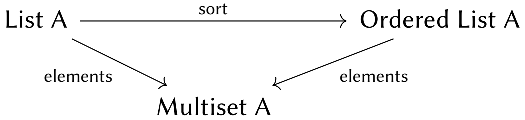

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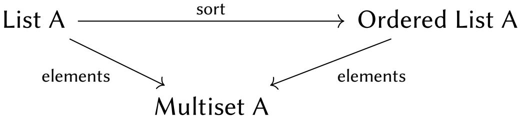

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$$\{g : \mathsf{FMSet}\ A\} \longrightarrow \mathsf{ElList}\ g \longrightarrow \mathsf{OElList}\ g$$

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Base Functors of Recursive Datatypes

- Recursive datatypes have a shape given by a **base functor** F
- E.g. Natural numbers: $(1 + -)$. Lists of element type A: $(1 + A \times -)$.
- Recursive datatype is given by fixpoint of composition of base functor F with itself
- Least fixpoint (μF): Inductive datatype. Greatest (νF): coinductive – not neccessarily well founded

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- Algebraically: fold alg where alg : F (Rec G) ⟶ Rec G
- Coalgebraically: unfold coalg where coalg : Rec F ⟶ G (Rec F)
- A way that gives us both...

Insertion- / Bubble Sort

(Hinze et al. 2012)

```
data L (r : Type) : Type where          -- aliasing
  []   : L r                            O = L
  _::_ : A → r → L r                    pattern _≤::_ x xs = x :: xs
```

```
swap : ∀ {x} → L (O x) → O (L x)
swap [] = []
swap (a :: []) =  a ≤:: []
swap (a :: (b ≤:: r)) with a ≤?≥ b
...| inl  a≤b  =  a  ≤:: ( b  ::  r )
...| inr  b≤a  =  b  ≤:: ( a  ::  r )
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insertSort = fold ( unfold  (swap ∘ L₁ out))
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The Finite Multiset Quotient Inductive Type

(Choudhury and Fiore 2023)

```
data FMSet (A : Type ℓ) : Type ℓ where
  []     : FMSet A
  _::_   : (x : A) → (xs : FMSet A) → FMSet A
  comm : ∀ {x y xs} → x :: y :: xs ≡ y :: x :: xs
  trunc  : isSet (FMSet A)
```

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1 :: 2 :: 3 :: [] ≡⟨ cong (1 ::_) comm ⟩ 1 :: 3 :: 2 :: []
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pattern []ℳ = []
pattern _::ℳ_ x xs = x :: xs
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Base Functors for (Ordered) Element-Indexed Lists

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data L   (r : Type)        :  Type           where
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data L   (r : Type)                  :  Type                  where
  []     : L r
  _::_   : A                         →  r          → L r

data L   (r : FMSet A → Type)        :  FMSet A → Type where
  []     : L r []𝓜
  _::_   : ∀ {g} → (x : A)           →  (r g)      → L r (x ::𝓜 g)
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Base Functors for (Ordered) Element-Indexed Lists

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data L  (r : Type)                 :  Type                where
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data L  (r : FMSet A → Type)       :  FMSet A → Type where
  []    : L r []M
  _::_  : ∀ {g} → (x : A)          →  (r g)               → L r (x ::M g)

data O  (r : FMSet A → Type)       :  FMSet A → Type where
  []    : O r []M
  _≤::_ : ∀ {g} (x : A) → (r g) → All (x ≤_) g → O r (x ::M g)
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The FMSet Index as a Termination Measure

L-coalgebras are well founded

```
pattern _::_^_ x xs g = _::_ {g = g} x xs

unfoldL : {r : FMSet A → Type} →
  ( ∀ {gᵣ } → (r  gᵣ ) → L r  gᵣ  ) → (∀ {g} → (r g) → ElList g)
unfoldL grow {_}              seed with  grow seed
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unfoldL grow  .{[]M}          _       |  []                  =
unfoldL grow  .{x ::M  g'}    _       |  x :: seed′ ^  g'    =
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- **with**-abstraction: Pattern matching refines earlier arguments, propagates information about the indexee back to the index

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unfoldL grow {_}            seed with grow seed
unfoldL grow .{[]ℳ}         _    | []                  = []
unfoldL grow .{x ::ℳ g'}    _    | x :: seed' ^ g'     =
                                   x :: unfoldL grow {g'} seed'
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- Index-preservation forces index of **seed** and **grow seed** to coincide
- **with**-abstraction: Pattern matching refines earlier arguments, propagates information about the indexee back to the index
- Index of recursive argument is smaller

Well Founded Recursion

- Syntactic termination checking based on dot-patterns of HITs inconsistent (Pitts 2020)

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- Define a family of maps indexed by FMSet A by well founded induction on the length of the index
- length defined by eliminating from FMSet A as the free commutative monoid to (ℕ , +) by λ a → 1

swap, Revisited

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  subst (O (L _)) comm
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- transpX-O (λ n → ...) i0 ... (Cavallo and Harper 2019)

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- Discard index with toList : {g : FMSet A} → OEIList g → List A

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- Intrinsically correct algorithms from correct distr. law
- For verified quick/treesort & heapsort following (Hinze et al. 2012), semantics via slice category $\rightarrow$ see paper

Future Work

- Conditions under which coalgebras are recursive in an indexed/fibered setting

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- More algorithms to verify with a distributive law as business logic