Natural Transformations as Business Logics: An Operational Intuition.

Cass Alexandru

2025-01-10

Motivation

- “Sorting with Bialgebras and Distributive Laws” (HJHWM, 2012)
- NLFPD ’24: I gave a talk abt bialgebraic semantics
- There: Post-hoc analysis of recursion behavior of sorting algs
- Contrived introductory examples: `length`, `replicate`
- This talk: Bottom-up operational intuition for distr. laws as business logic
- Revisit and uncontrive `length`, `replicate` examples
- Haskell examples again using the `recursion-schemes` library
- Less Category Theory, more TikZ animations

Structure

Base Functors of Recursive Datatypes

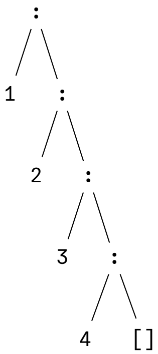
- Recursive datatypes have a shape given by a *base functor* F
- E.g. Natural numbers: $(1 + -)$. Lists of element type A: $(1 + A \times -)$.
- Recursive datatype is given by fixpoint of composition of base functor F with itself
- Least fixpoint: Inductive datatype. Greatest: coinductive – not nec. well founded

Induction

- Can *eliminate* (map out of) a recursive datatype
- Algebra: Compositionally interpret *syntax* to a domain by giving it *semantics*, giving each constructor an *interpretation*.
- Roll up the datatype from the base cases (Bottom-up).

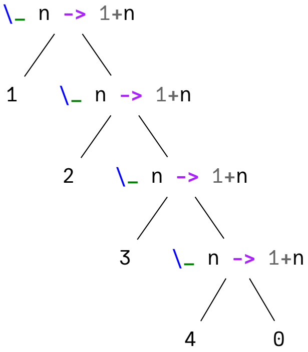
Functions Replace Constructors

List

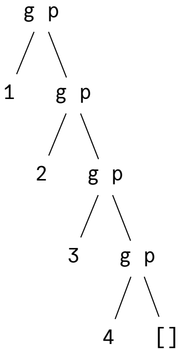


Traversals

`length`



`filter p`




```
g p x xs =
  (if p x then [x] else [])
  ++ xs
```

Example Evaluation of filter even

```
g even x xs =
  (if even x then [x] else []) ++ xs
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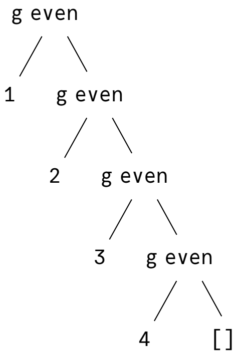

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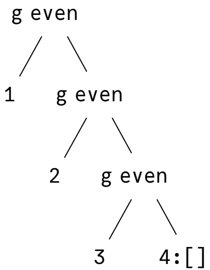

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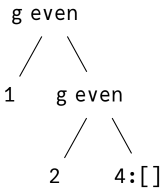

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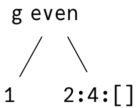

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```
2:4:[]
```

Coinduction

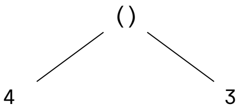
- Map *into* a recursive datatype
- Coalgebra: From a seed, create one level of the datatype, with new seeds at recursive positions
- Iteratively apply until base cases are reached
- NB: Base cases may not be reached! $\rightarrow$ non well founded trees

Example: Growing a Fibonacci Tree

5

```
fib :: Nat -> TreeF () Nat
fib = \case
  0 -> NodeF () []
  1 -> NodeF () []
  n -> NodeF () [n-1,n-2]
```

Example: Growing a Fibonacci Tree

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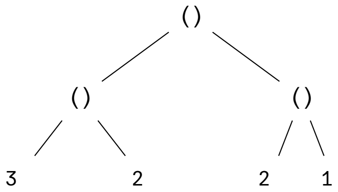


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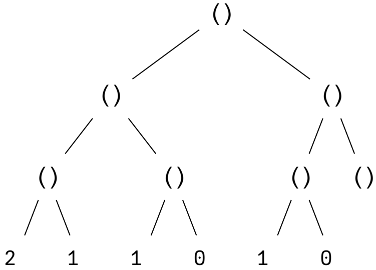


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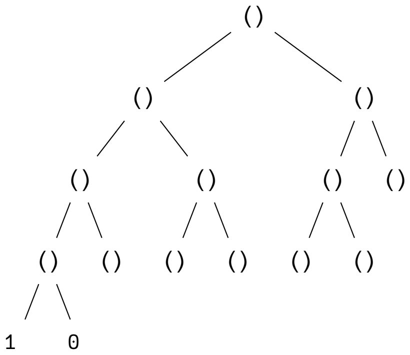


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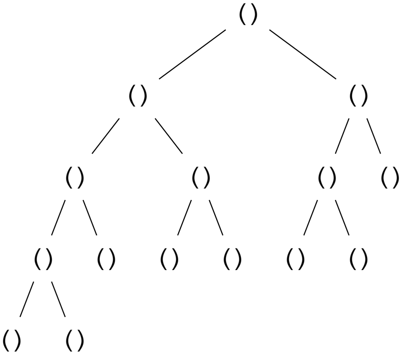


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```
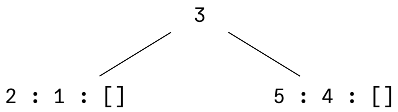

Example: Growing a BST with partition

```
2 : 5 : 4 : 1 : 3 : []
```

```haskell
partition :: (Ord a) =>
  [a] -> (TreeF a) [a]
```

Example: Growing a BST with `partition`

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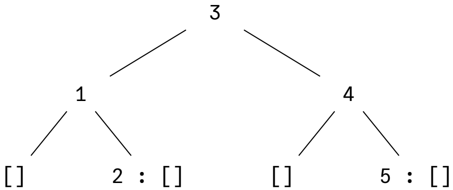


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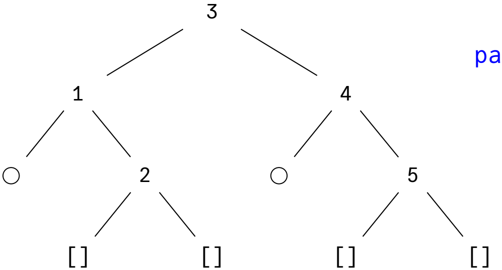


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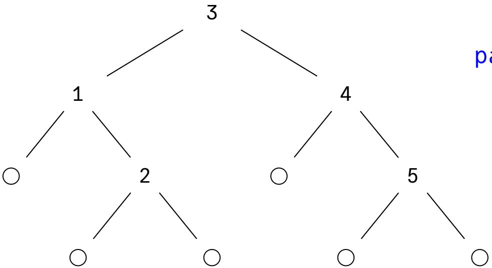


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Structure

Maps Between Recursive Datatypes

- `Rec F -> Rec G`
- Algebraically: `F (Rec G) -> Rec G`
- Coalgebraically: `Rec F -> G (Rec F)`
- A secret third option?

Natural Transformations

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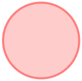
- 
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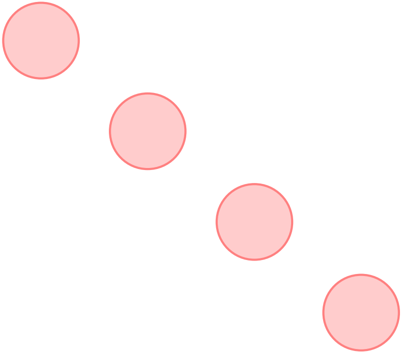
```
-

length & replicate

```haskell
forget :: ListF a :=> NatF
forget = \case
  Nil -> Zero
  Cons _ x -> Suc x
```

```haskell
deco :: a -> NatF :=> (ListF a)
deco e = \case
  Zero -> Nil
  Suc x -> Cons e x
```

Bottom-up



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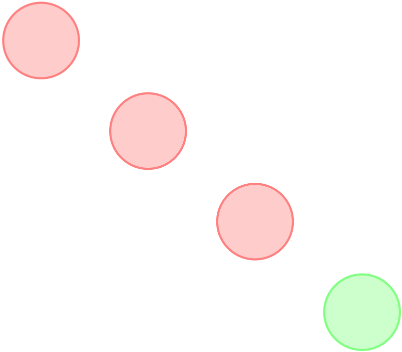

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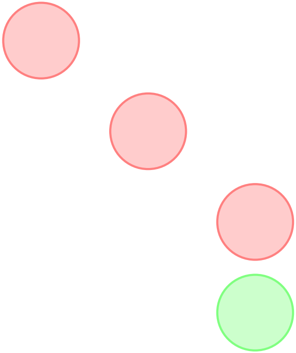

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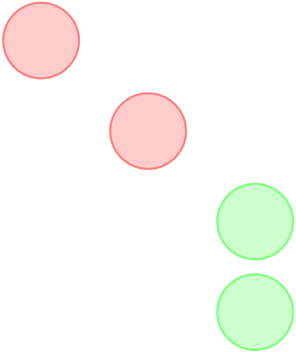

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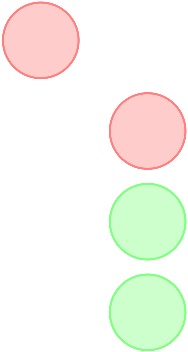

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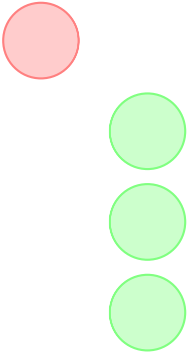

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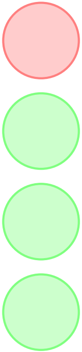

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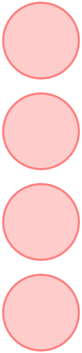

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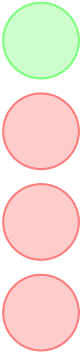

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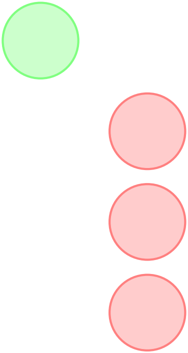

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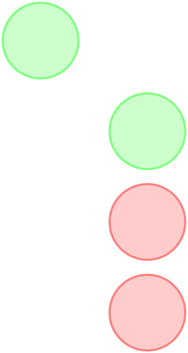

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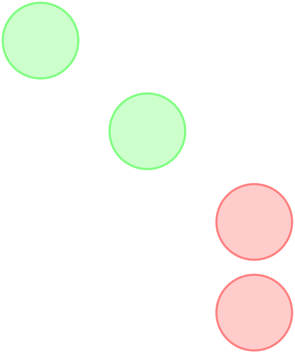

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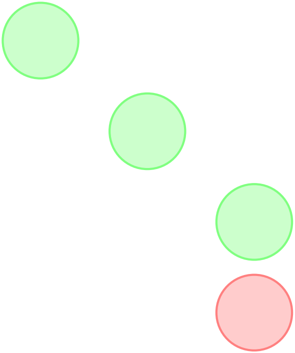

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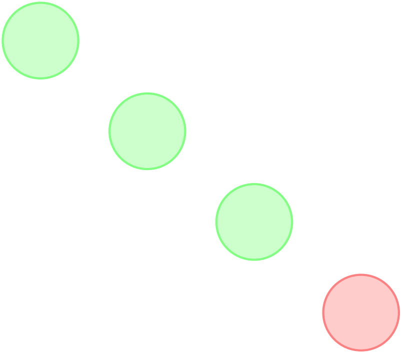

Top-down

Natural Transformation Semantics

```
natSem :: forall μf νg . (Recursive μf, Corecursive νg) =>
  (Base μf :=> Base νg) -> μf -> νg
natSem δ = fold @μf alg where
   alg :: (Base μf) νg -> νg
   alg = embed @νg . δ

coNatSem :: forall μf νg . (Recursive μf, Corecursive νg) =>
  (Base μf :=> Base νg) -> μf -> νg
coNatSem δ = unfold @νg coalg where
  coalg :: μf -> (Base νg) μf
  coalg = δ @μf . project
```

- NB: For `natSem`, we used `embed` from the `recursion-schemes` library
- Corresponds to the initial algebra ⇒ `νg` isn't actually codata
- For `coNatSem` we used `unfold`. But (`δ @μf . project`) is a *recursive coalgebra* (transposition proposition in (Eppendahl, 2000))
- Both of these semantics go between carriers of *initial* algebras, so data, not codata

Structure

Distributive Laws

- $FG \Rightarrow GF$

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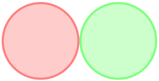
- 
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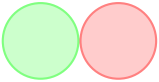
- 

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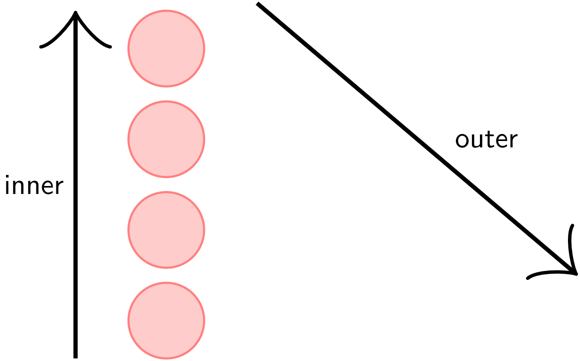

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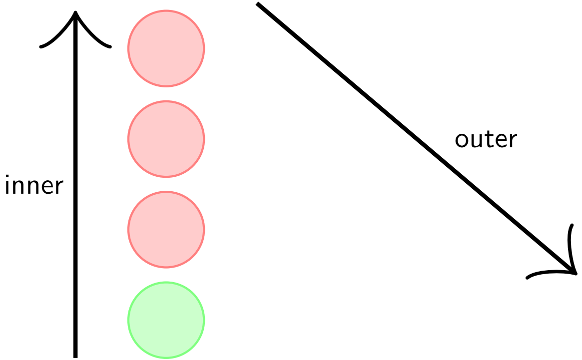

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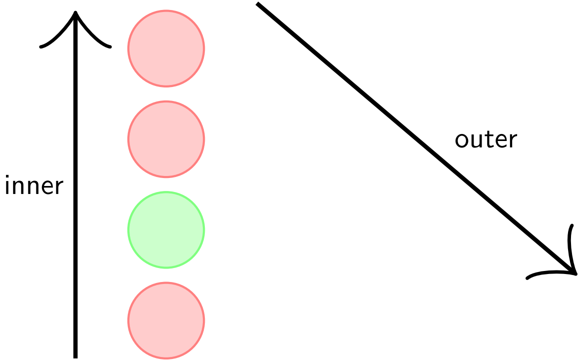

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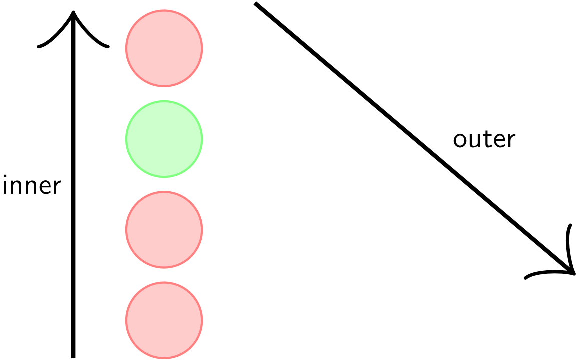

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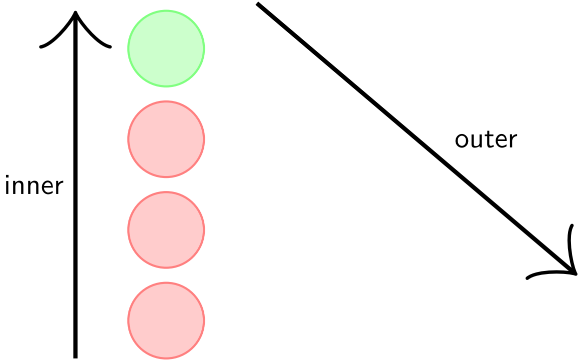

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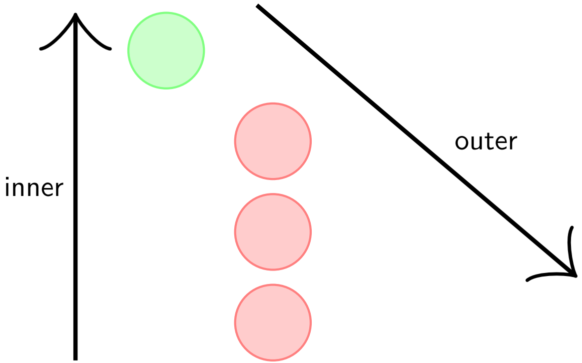

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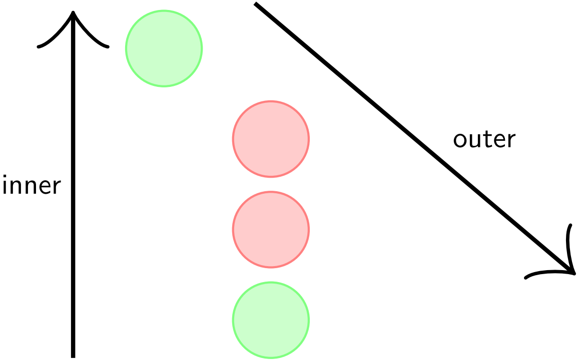

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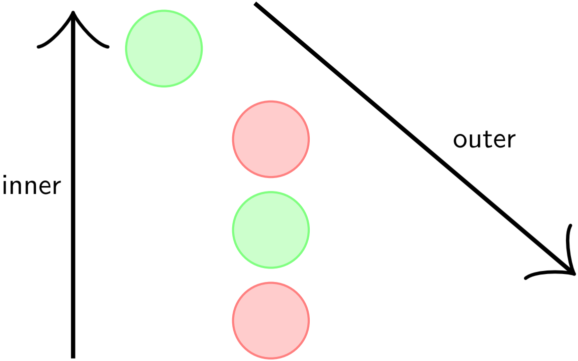

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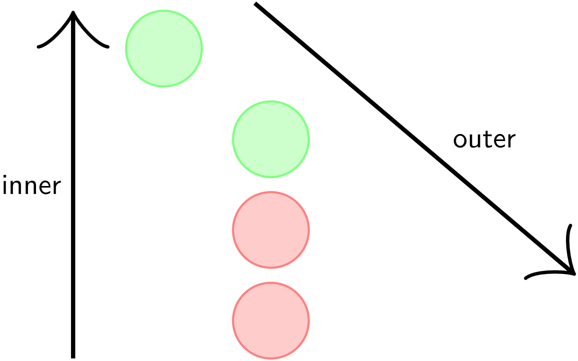

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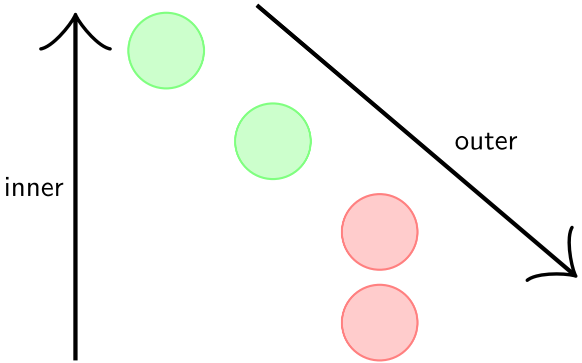

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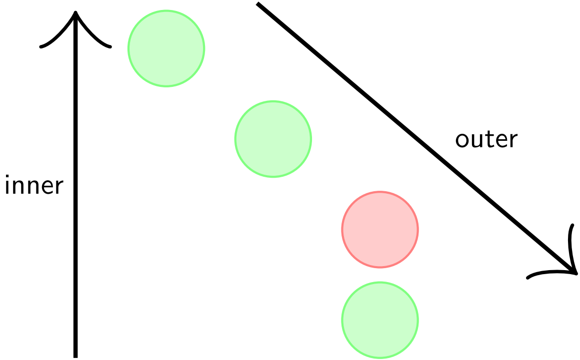

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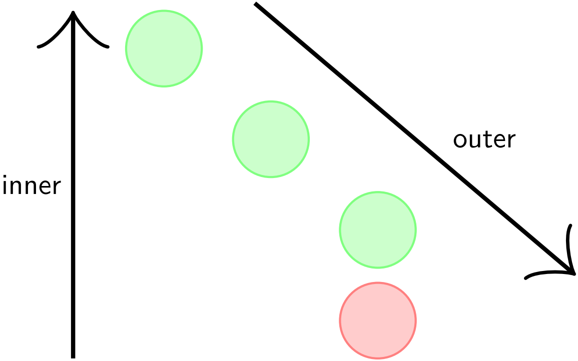

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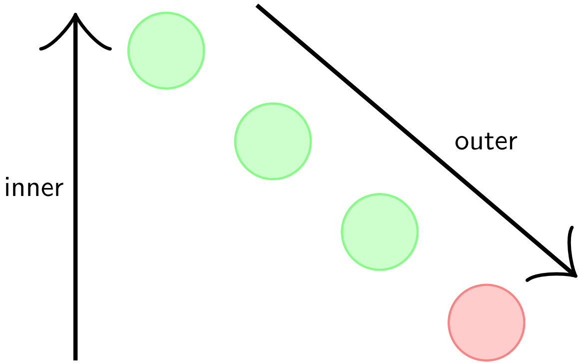

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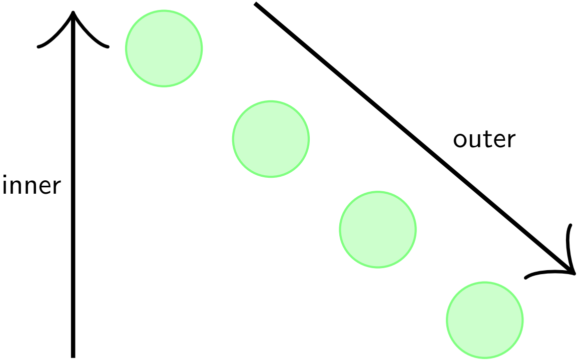

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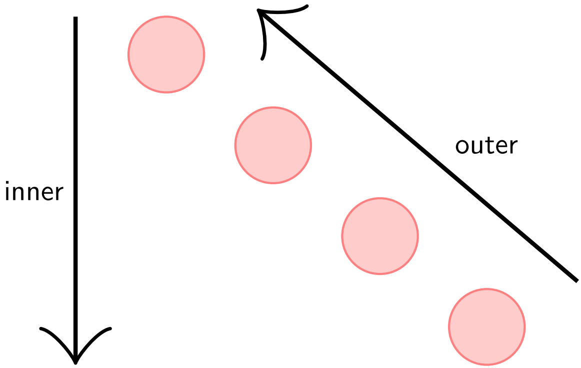

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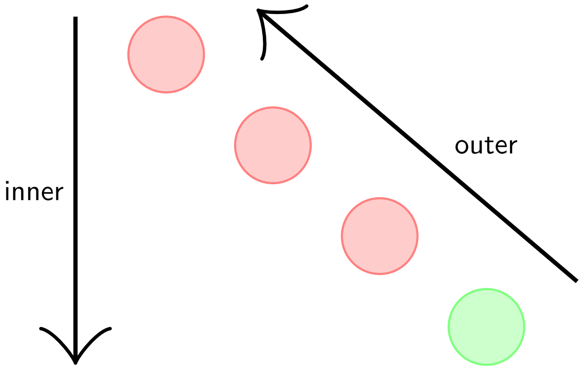

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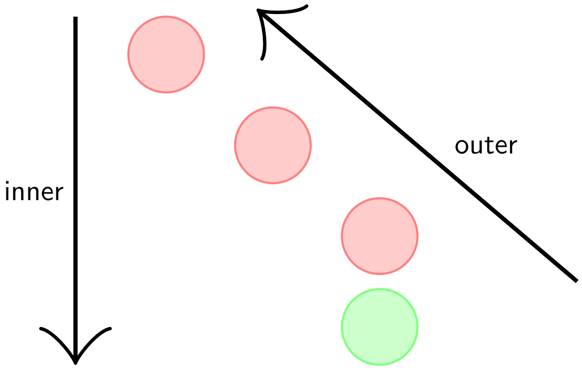

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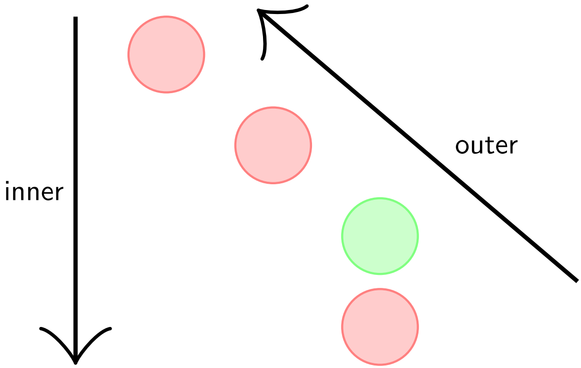
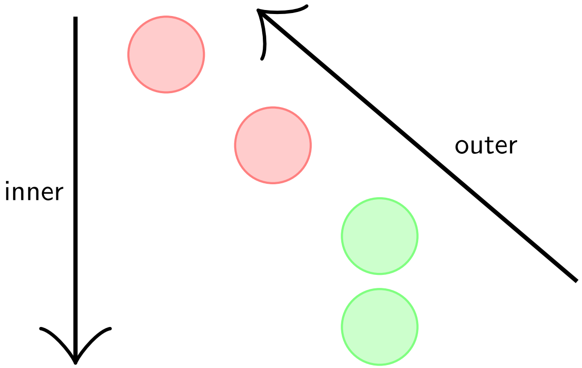


Bottom-up 2



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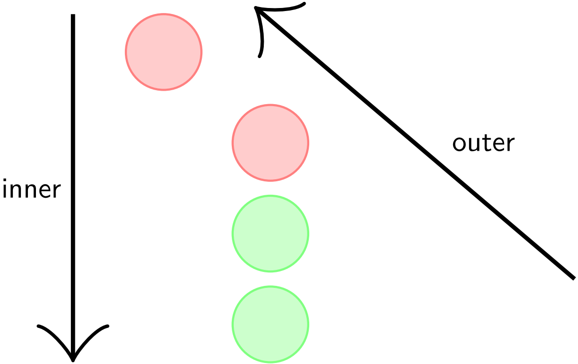

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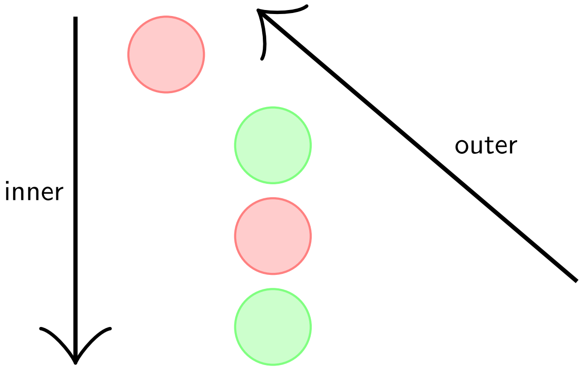

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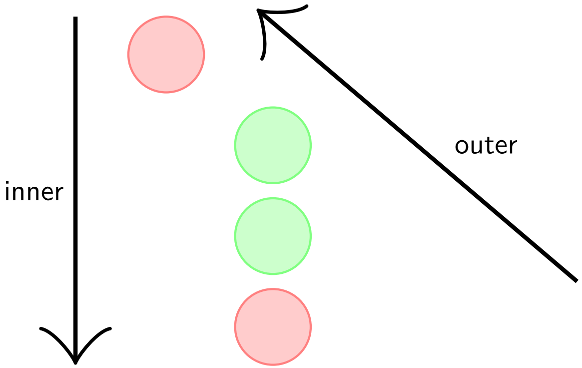

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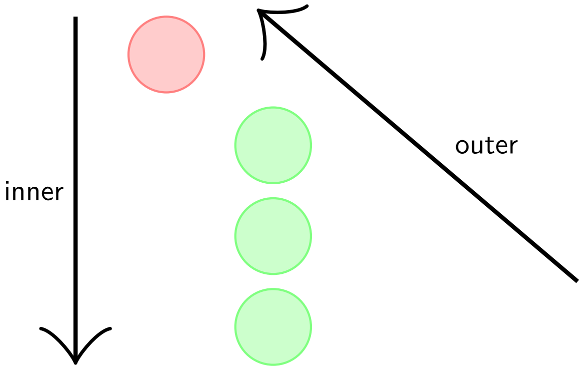

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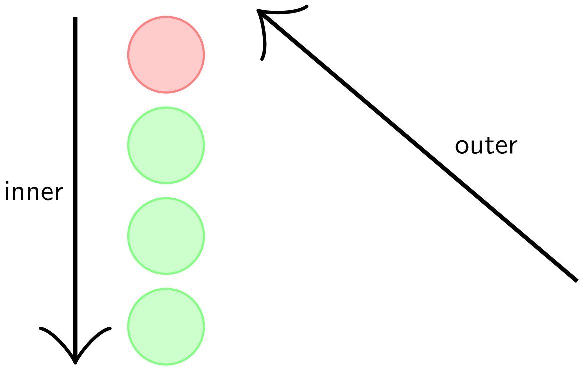

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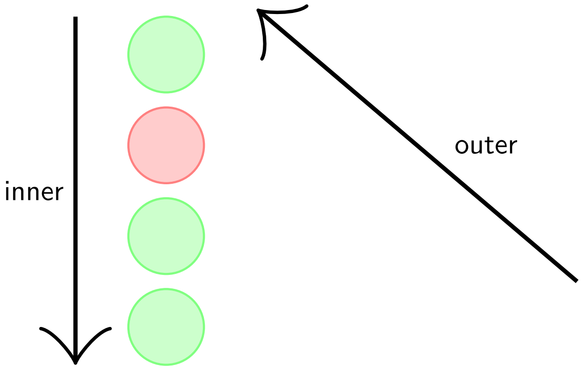

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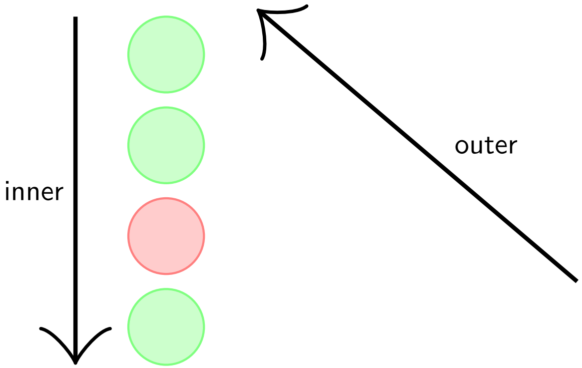

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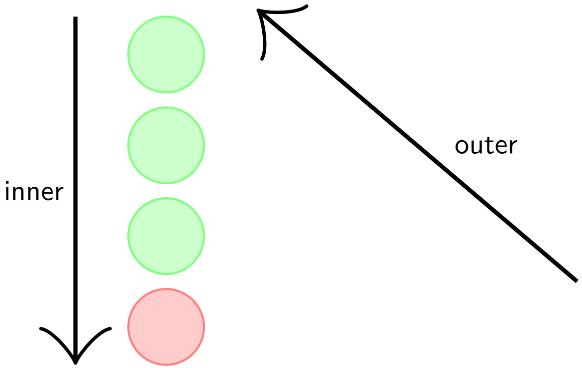

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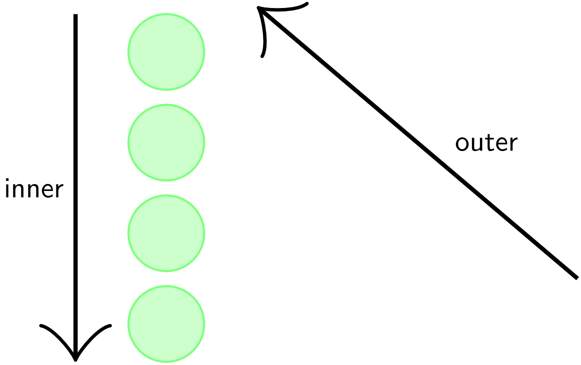
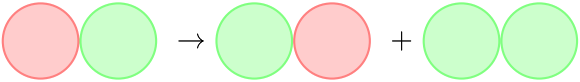


Insights

- Early termination avails only in the nested coalgebraic step
- 
- Lazy evaluation: Variant with outer unfold is always *incremental*, outer fold is (in general) *monolithic*

Example: Insertion- / Selection Sort

```
σ :: (Ord a) => L a (O a r) -> Either (O a (L a r)) (O a (O a r))
σ = \case
  Nil             -> Nil
  a `Cons` Nil    -> a `OCons` Nil
  a `Cons` (b `OCons` r)
    | a <= b      -> Right $ a `OCons` b `OCons` r
    | otherwise   -> Left  $ b `OCons` a `Cons` r
```

Applications

- Recent paper "Intrinsically Correct Sorting in Cubical Agda" (Alexandru and Choudhury and Rot and van der Weide, CPP '25)
- more in-depth on verified bialgebraic sorting algorithms
- encoding invariants in base functors led to the coincidence of partial and total correctness I talked about at NLFPD '24
- makes the input/output base functors meaningfully different (otherwise just aliased)
- correctness of BL in form of distr. law yields correctness of entire algorithm (both variants)

Structure

The Cata is a Lie

- $$\begin{array}{ccc} F\mu F & \xrightarrow{F?} & FX \\ \Big\downarrow \text{in} & & \Big\downarrow a \\ \mu F & \xrightarrow{?} & X \end{array}$$

The Cata is a Lie

- $$\begin{array}{ccc} F\mu F & \overset{F?}{\dashrightarrow} & FX \\ {\scriptstyle \mathrm{in}^{-1}}\big\uparrow\big\downarrow{\scriptstyle \mathrm{in}} & & \big\downarrow{\scriptstyle a} \\ \mu F & \overset{?}{\dashrightarrow} & X \end{array}$$

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- $? := a \circ F? \circ \mathrm{in}^{-1}$
- Coalgebra-to-algebra morphism from recursive coalgebra in^{-1}
- “We believe that, as long as structured recursion is concerned, recursive coalgebras are a more basic concept than initial algebras” – (Capretta and Uustalu and Vene, 2004)

Future Work

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- Termination checking in general is non-compositional and syntactic – I'm hoping to look into if rec. coalgs can change that
- More future work: More algorithms with BL in form of nat. trans. – simple, distr. law, or: distr. law with coherence conditions (comonad over a functor, etc.) (Turi and Plotkin, '97)

- `@cxandru@types.pl`
- `cxandru.ee`